Unit I : linear Differential equations with constant coefficients.

Differential equations are widely used in field of Engg. & Applied sciences. mathematical formulations of most of the physical problems are in the form of differential equations. use of differential equations is most prominent in subjects like circuit Analysis, theory of structure Vibration, Heat ~~transform~~, transfer fluid mechanics etc.

second order polynomials and their Factorizations

$D^2-2D-3 = (D+1)(D-3)$ $D^2-4D+4 = (D-2)^2$

$D^2+5D+6 = (D+2)(D+3)$ $D^2-a^2 = (D-a)(D+a)$

$D^2+2D+1 = (D+1)^2$ $D^2+a^2 = (D+ia)(D-ia)$

$D^2-5D+6 = (D-2)(D-3)$

$D^2+3D+2 = (D+2)(D+1)$

$D^2-D-2 = (D-2)(D+1)$

If the roots of $ax^2+bx+c=0$, are $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

these roots are imaginary if $b^2-4ac < 0$

$D^2+2D+2=0, \Rightarrow D = \frac{-2\pm\sqrt{4-8}}{2} = -1 \pm i$

$D^2+D+1=0 \Rightarrow D = \frac{-1\pm\sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{3}}{2} i$

If $D = \frac{-1}{2} \pm i\frac{\sqrt{3}}{2} = \alpha \pm i\beta$ then $\alpha = \frac{-1}{2}$ $\beta = \frac{\sqrt{3}}{2}$. β is always positive, α may be positive, negative or zero.

$D^2+1=0 \Rightarrow D^2=-1$ ie $D=\pm i$ $\therefore \alpha=0, \beta=1$

$D^2+4=0 \Rightarrow D^2=-4$ $D=\pm 2i$ $\alpha=0, \beta=2$.

Third degree polynomial & Their factorization

$D^3-a^3 = (D-a)(D^2+aD+a^2)$ $D^3+3D^2+3D+1 = (D+1)^3$

$D^3+a^3 = (D+a)(D^2-aD+a^2)$ $D^3-3D^2+3D-1 = (D-1)^3$

Use of synthetic divison method

① $f(D) = D^3 - 7D - 6 = 0$, for $D = -1$, $f(-1) = 0$.

$\therefore$ $D+1$ is one of the factor.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$\therefore$ $D^3 - 7D - 6 = 0 \Rightarrow (D+1)(D^2 - D - 6) = 0$

$\Rightarrow (D+1)(D-3)(D+2) = 0$

$\Rightarrow D = -1, -2, 3.$

② for $D^3 - 2D + 4 = 0$, $D = -2$, $f(-2) = 0$

$(D+2)$ is one of the factor

$$\begin{array}{r|rrrr} -2 & 1 & 0 & -2 & 4 \\ & & -2 & 4 & -4 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$D^3 - 2D + 4 = 0 \Rightarrow (D+2)(D^2 - 2D + 2) = 0$

$\Rightarrow (D+2) = 0$ & $D = 1 \pm i$

Fourth Degree polynomial and their Factorization

(a) $D^4 - a^4 = (D^2 - a^2)(D^2 + a^2) = (D-a)(D+a)(D+ia)(D-ia)$

(b) making perfect square by introducing middle term

for $D^4 + a^4 = 0$ consider $(D^2 + a^2)^2 = D^4 + 2a^2D^2 + a^4$

$D^4 + a^4 = D^4 + 2a^2D^2 + a^4 - 2a^2D^2$

$= (D^2 + a^2)^2 - (\sqrt{2}\, aD)^2$

$= (D^2 - \sqrt{2}\, aD + a^2)(D^2 + \sqrt{2}\, aD + a^2)$

for $D^4 + 1 = D^4 + 2D^2 + 1 - 2D^2$

$= (D^2 + 1)^2 - (\sqrt{2} D)^2$

$= (D^2 + \sqrt{2} D + 1)(D^2 - \sqrt{2} D + 1)$

(c) $D^4 + 8D^2 + 16 = (D^2 + 4)^2$

$D^4 + 2D^2 + 1 = (D^2 + 1)^2 = (D+i)^2(D-i)^2$

$D^4+10D^2+9 = (D^2+9)(D^2+1) = (D+3i)(D-3i)(D+i)(D-i)$

The n^{th} order linear differential equation with constant coefficients

A differential equation which contains the differential coefficients and the dependent variable in the first degree, doesnot involve the product of derivative with another derivative or dependent variable, and in which the coefficients are constants is called a linear differential equation with constant coefficients.

A general form of such differential equation of order "n" is

$$a_0\frac{d^ny}{dx^n} + a_1\frac{d^{n-1}y}{dx^{n-1}} + a_2\frac{d^{n-2}y}{dx^{n-2}} + \cdots\cdots + a_{n-1}\frac{dy}{dx} + a_ny = f(x) \quad -①$$

Here $a_0, a_1, a_2, \ldots, a_n$ are constants, Equation ① is n^{th} order linear differential equation with constant coefficients.

eg. put $n=2$ in equation (1) we get $a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = f(x)$ which is second order ~~di~~ linear differential equation with constant co-efficients

Using the differential operator $D = \frac{d}{dx}$ ie $D^2 = \frac{d^2}{dx^2} \cdots$ $D^n = \frac{d^n}{dx^n}$, equation ① will take the form.

$$a_0D^ny + a_1D^{n-1}y + a_2D^{n-2}y + \cdots\cdots + a_{n-1}Dy + a_ny = f(x)$$

or

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \cdots + a_{n-1}D + a_n)y = f(x) \quad --②$$

in which each term in the bracket is operating on y and the results are added.

let $\phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$, $\phi(D)$ is called n^{th} order polynomial in D.

$\therefore$ Eqn (2) can be written as $\boxed{\phi(D)y = f(x)}$ (3)

The Nature of Differential operator

It is convenient to introduce the symbol D to represent the operation of differentiation w.r.t. x ie $D = \frac{d}{dx}$ so that

$$\frac{dy}{dx} = Dy, \quad \frac{d^2y}{dx^2} = D^2y, \cdots \quad \frac{d^ny}{dx^n} = D^n y \text{ and}$$

$$\frac{dy}{dx} + ay = (D+a)y.$$

The differential operator D obeys the laws of Algebra.

If y_1 & y_2 are differentiable functions of x, and a is constant and m, n are positive integers then

i) $D^m(D^n)y = D^n(D^m y) = D^{m+n} y$

ii) $(D-m_1)(D-m_2)y = (D-m_2)(D-m_1)y$

iii) $(D-m_1)(D-m_2)y = [D^2 - (m_1+m_2)D + m_1 m_2]y$

iv) $D(au) = aD(u)$, $D^n(au) = a D^n u$

v) $D(y_1+y_2) = Dy_1 + Dy_2$

vi) $D^n(y_1+y_2) = D^n y_1 + D^n y_2$.

Linear Differential equation $\phi(D)y = 0$

consider $\phi(D)y = 0$.

Where $\phi(D) = a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_n$ is nth order polynomial in D & D obeys the laws of algebra, we can in general factorize $\phi(D)$ in n linear factors as $\phi(D) = (D-m_1)(D-m_2) \cdots$

$\cdots (D-m_n)$. where $m_1, m_2, m_3 ---- m_n$ are the roots of the algebraic equation $\phi(D)=0$.

$\therefore$ equation (4) can be written as

$$\phi(D)y = (D-m_1)(D-m_2)(D-m_3) ---- (D-m_n)y = 0 \quad (5)$$

Note: These factors can be Taken in any sequence.

Auxiliary Equation (A.E)

The equation $\phi(D)=0$ is called as an auxiliary equation for equations (3), (4).

eg. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$

By using operator D for $\frac{d}{dx}$ We have

$$(D^2-5D+6)y = 0.$$

$\therefore$ $\phi(D) = D^2-5D+6 = 0$ is the A.E.

Solution of $\phi(D)y=0$

Being n^{th} order D.E. equation (4) or (5) will have exactly n arbitrary constants in its general solution.

eq^n (5) will be satisfied by the solution of the equation $(D-m_n)y=0$ ie $\frac{dy}{dx} - m_n y = 0$.

On solving this 1^{st} order & 1^{st} degree diff. eq^n by separating variables, we get $y = C_n e^{m_n x}$

where C_n is arbitrary constant.

Similarly, since the factors in equation (5) can be taken in any order, the equation will be satisfied by the solution of each of the equations $(D-m_1)y=0$, $(D-m_2)y=0$ .... etc. that is by $y=c_1e^{m_1x}$, $y=c_2e^{m_2x}$ .... etc.

It can be easily proved that the sum of these individual solutions ie

$$\boxed{y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}} \quad (6)$$

also satisfies eqn (5), and it contains n-arbitrary constants and (4) is of the n^{th} order, (6) constitute the general solution of the eqn (4).

$\therefore$ The general solution of $\phi(D)y=0$ is

$$y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}$$

Where $m_1, m_2, m_3 \cdots m_n$ are the roots of the A.E. $\phi(D)=0$.

Different cases depending upon the Nature of Roots of the Auxiliary Equation $\phi(D)=0$.

A. Case of Real & Different Roots

If the roots of $\phi(D)=0$ be $m_1, m_2, m_3 \cdots m_n$ all are real & different, then the solution of $\phi(D)y=0$ will be

$$\boxed{y = c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_ne^{m_nx}}$$

Where $c_1, c_2, c_3 \cdots c_n$ are arbitrary constants.

Case of Real & Repeated Roots

If $m_1 = m_2$ are real, and the remaining roots m_3, m_4, m_5 ...... m_n are real & different then the solution of $\phi(D)y = 0$ is

$$y = (c_1x + c_2)e^{m_1x} + c_3e^{m_3x} + c_4e^{m_4x} + \cdots + c_ne^{m_nx}.$$

Similarly if three roots are repeated ie if $m_1 = m_2 = m_3$ are real and the remaining roots $m_4, m_5, \cdots m_n$ are real & distinct, the the solution of $\phi(D)y = 0$ is

$$y = (c_1x^2 + c_2x + c_3)e^{m_1x} + c_4e^{m_4x} + c_5e^{m_5x} + \cdots + c_ne^{m_nx}.$$

If $m_1 = m_2 = m_3 = \cdots m_n$ ie n roots are real and equal then the solution of $\phi(D)y = 0$ is

$$\cancel{y = (c_1e^{m_1x} + c_2e^{m_2x} + c_3e^{m_3x} + \cdots + c_n}$$

$$y = (c_1x^{n-1} + c_2x^{n-2} + c_3x^{n-3} + \cdots + c_n)e^{m_1x}$$

Ex 1 For $(D^2 - 6D + 9)y = 0$. A.E $= (D-3)^2 = 0$ &

sol^n is $y = (c_1x + c_2)e^{3x}$.

2. for $(D-1)^3(D+1)y = 0$ sol^n is $y = (c_1x^2 + c_2x + c_3)e^{x} + c_4e^{-x}$

3. for $(D-1)^2(D+1)^2y = 0$ sol^n is

$$y = (c_1x + c_2)e^{x} + (c_3x + c_4)e^{-x}.$$

solve ①

c) Case of imaginary (complex) Roots

If the roots of A.E. $\phi(D)=0$ are imaginary ie if $\alpha \pm i\beta$ be one such pair, then, $m_1 = \alpha+i\beta$, $m_2=\alpha-i\beta$

The part of the soln corresponding to these roots takes the form

$$y = Ae^{(\alpha+i\beta)x} + Be^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left[Ae^{i\beta x} + Be^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[A(\cos\beta x + i\sin\beta x) + B(\cos\beta x - i\sin\beta x)\right]$$

$$= e^{\alpha x}\left[(A+B)\cos\beta x + i(A-B)\sin\beta x\right]$$

$$y = e^{\alpha x}\left[c_1\cos\beta x + c_2\sin\beta x\right]$$

Where $c_1 = A+B$, $c_2 = i(A-B)$ are arbitrary constants

Ex 1) solve $(D^2+2D+5)y=0$

Ans: $D=-1\pm 2i$, $y = e^{-x}[A\cos 2x + B\sin 2x]$

2) $(D^2+4)y=0$,

Ans: $D=\pm 2i$, $y = A\cos 2x + B\sin 2x$.

D) Case of Repeated imaginary roots

If the imaginary roots $m_1 = \alpha+i\beta$ & $m_2=\alpha-i\beta$ occours twice, then the part of the solution ~~$\phi(D)$~~ of $\phi(D)y=$ will be

$$y = (Ax+B)e^{m_1 x} + (Cx+D)e^{m_2 x}$$

$$= (Ax+B)e^{(\alpha+i\beta)x} + (Cx+D)e^{(\alpha-i\beta)x}$$

$$= e^{\alpha x}\left[(Ax+B)e^{i\beta x} + (Cx+D)e^{-i\beta x}\right]$$

$$= e^{\alpha x}\left[(Ax+B)(\cos\beta x + i\sin\beta x)\} + (Cx+D)\{\cos\beta x - i\sin\beta x\}\right\}$$

$$= e^{\alpha x}\left[(Ax+B+Cx+D)\cos\beta x + i(Ax+B-Cx-D)\sin\beta x\right\}$$

$$\boxed{y = e^{\alpha x}\left[(c_1x+c_2)\cos\beta x + (c_3x+c_4)\sin\beta x\right]}$$

Solve $(D^4+2D^2+1)y=0$.

A.E is $D^2+2D^2+1=0$. $\Rightarrow D=\pm i \pm i$

roots are repeated 2-times Hence sol^n is

$$y = e^{0x}\left([c_1x+c_2)\cos x + (c_3x+c_4)\sin x\right]$$

Solve $\frac{d^4y}{dx^4} - 16y = 0$.

Ans: Given $(D^4-16)y=0$. A.E is $D^4-16=0$

$$\Rightarrow (D^2-4)(D^2+4)=0$$

$$\Rightarrow (D-2)(D+2)(D+i2)(D-i2)$$

$$\Rightarrow D=2,-2,\quad D=-2i \pm 2i$$

$$y = c_1e^{2x} + c_2e^{-2x} + c_3\cos 2x + c_4\sin 2x$$

Exercise

A) solve the following differential equations

1) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} - 6y = 0.$ $D = -1, 6.$ $y = c_1 e^{-x} + c_2 e^{6x}$

2) $2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 10y = 0.$ $D = -2, \frac{5}{2}.$ $y = c_1 e^{-2x} + c_2 e^{\frac{5}{2}x}$

3) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0.$ $D = 0, -1, -1.$ $y = c_1 + e^{-x}(c_1 x + c_2)$

4. $(D^4 - 2D^3 + D^2)y = 0$ $D = 0, 0, 1, 1.$

$y = c_1 x + c_2 + e^{x}(c_3 x + c_4)$

5. $\frac{d^4y}{dx^4} - 16y = 0.$ $D = 2, -2, \pm 2i$

$y = c_1 e^{2x} + c_2 e^{-2x} + (c_3 \cos 2x + c_4 \sin 2x)$

6. $4y'' - 8y' + 7y = 0.$ $D = 1 \pm \frac{\sqrt{3}}{2} i$

$y = e^{x}[A \cos \frac{\sqrt{3}}{2}x + B \sin \frac{\sqrt{3}}{2}x]$

7. $(D^3 + D^2 - 2D + 12)y$ $D = -3, 1 \pm \sqrt{3}i$

$y = e^{-3x} + e^{x}[A \cos\sqrt{3}x + B \sin\sqrt{3}x]$

8. $(D^4 + 2D^2 + 1)y = 0$ $D = \pm i, \pm i.$

$y = (c_1 x + c_2)\cos x + (c_3 x + c_4)\sin x$

9. $(D^2+1)^3(D^2+D+1)^2 y = 0.$ $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x$

$$+ e^{-\frac{x}{2}}\left[(c_7+c_8x)\cos\left(\frac{\sqrt{3}}{2}x\right) + (c_9+c_{10}x)\sin\frac{\sqrt{3}}{2}x\right]$$

10. $\frac{d^4y}{dx^4} + m^4y = 0.$

Sol: Given $(D^4+m^4)y=0.$

A.E is $D^4+m^4=0. \Rightarrow D^4+2D^2m^2+m^4-2D^2m^2=0$

$$\Rightarrow (D^2+m^2)^2-(\sqrt{2}Dm)^2=0$$

$$\Rightarrow (D^2-\sqrt{2}Dm+m^2)(D^2+\sqrt{2}mD+m^2)=0$$

$\Rightarrow D^2-\sqrt{2}mD+m^2=0$ & $D^2+\sqrt{2}mD+m^2=0$

$$\therefore D = \frac{\sqrt{2}m \pm \sqrt{2m^2-4m^2}}{2}$$

$$= \frac{\sqrt{2}m \pm \sqrt{-2m^2}}{2} = \frac{\sqrt{2}m \pm \sqrt{2}mi}{2}$$

$$= \frac{m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}i$$

$$\Rightarrow D = \frac{-m}{\sqrt{2}} \pm \frac{m}{\sqrt{2}}i$$

$$\therefore y_c = e^{\frac{m}{\sqrt{2}}x}\left[c_1\cos\frac{m}{\sqrt{2}}x + c_2\sin\frac{m}{\sqrt{2}}x\right] + e^{\frac{-m}{\sqrt{2}}x}\left[c_3\cos\frac{m}{\sqrt{2}}x + c_4\sin\frac{m}{\sqrt{2}}x\right]$$

11. $\frac{d^4y}{dx^4} - 5\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 28y = 0$

$$D = -2, -2, 2 \pm \sqrt{3}i$$

$$y = (c_1x+c_2)e^{-2x} + e^{2x}[A\cos\sqrt{3}x + B\sin\sqrt{3}x]$$

The General solution of the linear differential eq^n $\phi(D)y = f(x)$

The general solution of the equation $\phi(D)y = f(x)$ can be written as $y = y_c + y_p$ where

1) y_c is the solution of the given equation with $f(x) = 0$, that is of the equation $\phi(D)y = 0$ (which is known as associated equation or reduced equation) and is called the complementary ~~equation~~ function (C.F). It involves n arbitrary constants and is denoted by C.F. then

$$\boxed{\phi(D)y_c = 0}$$

2. y_p is any function of x, which is satisfies the equation $\phi(D)y = f(x)$, so that $\boxed{\phi(D)y_p = f(x)}$

y_p is called the particular integral and is denoted by P.I. It doesnot contain any arbitrary constant.

Thus on substituting $y = y_c + y_p$ in $\phi(D)y$

$$\phi(D)[y_c + y_p] = \phi(D)y_c + \phi(D)y_p$$
$$= 0 + f(x)$$
$$= f(x)$$

$\therefore$ $y = y_c + y_p$ satisfies the equation $\phi(D)y = f(x)$ & as it contains exactly n arbitrary constants, is the general solution (complete) solution of the eq^n.

Note 1: The complete sol of $\phi(D)y = f(x)$ is $y = y_c + y_p = C.F + P.I.$

2. The general solution of $\phi(D)y = f(x)$ has arbitrary constants equal ~~to~~ ~~th~~ in number to the order of the differential equation.

Symbolic Representation of P.I

The P.I. of $\phi(D)y = f(x)$ is denoted by y_p or P.I & is given by

$$P.I = y_p = \frac{1}{\phi(D)} f(x)$$

eg. 1) for $(D^2-1)y = x^2$, $\quad y_p = \frac{1}{D^2-1} x^2$.

2) For $(D^3+2D^2+1)y = e^x$, $\quad y_p = \frac{1}{D^3+2D^2+1} e^x$.

Methods of obtaining particular ~~me~~ Integral

There are three methods to evaluate the particular integral $y_p = \frac{1}{\phi(D)} f(x)$.

A. General method

B. short cut methods

C. Method of Variation of parameters.

A. General method

This method is useful when the shortcut methods given in B are not applicable. This method involves integration.

1) $\frac{1}{D-m} f(x)$: By definition of P.I.. $\frac{1}{D-m} f(x)$ will be the P.I of the equation $(D-m)y = f(x)$ ie the part in the solution of this equation which doenot contain the arbitrary constant. We have,

$$\frac{dy}{dx} - my = f(x) \qquad \text{(linear)}$$

I.F $= e^{-mx}$ & the general sol^n is

$$y e^{-mx} = \int f(x) \cdot e^{-mx} dx + c_1$$

$$y = c_1 e^{mx} + e^{mx} \int f(x) e^{-mx} dx$$

$$= y_c + y_p$$

Hence $c_1 e^{mx}$ is the C.F. and $e^{mx} \int f(x) e^{-mx} dx$ must be P.I.

$$\boxed{y_p = P.I = \frac{1}{D-m} f(x) = e^{mx} \int e^{-mx} \cdot f(x) dx}$$

Similarly $\boxed{y_p = P.I = \frac{1}{D+m} f(x) = e^{-mx} \int e^{mx} f(x) dx}$

put $m=0$, $y_p = \frac{1}{D} f(x) = \int f(x) dx$

Also

$$y_p = \frac{1}{D^2} f(x) = \frac{1}{D}\left[\frac{1}{D} f(x)\right] = \frac{1}{D}\left[\int f(x) dx\right]$$

$$= \int \left[\int f(x) dx\right]$$

Similarly

$$y_p = \frac{1}{D^3} f(x) = \int \left\{ \int \left[\int f(x) dx \right] \right\} \text{ and so on.}$$

(ii) $\frac{1}{(D-m_1)(D-m_2)} f(x)$:

$$y_p = \frac{1}{(D-m_1)(D-m_2)} f(x) = \frac{1}{D-m_1} \left[e^{m_2 x} \int e^{-m_2 x} f(x) dx\right]$$

$$= e^{m_1 x} \int e^{-m_1 x} \cdot \left[e^{m_2 x} \int e^{-m_2 x} f(x) dx\right] dx$$

iii) Use of partial fraction

$$y_p = \frac{1}{(D-m_1)(D-m_2)} f(x) = \frac{1}{m_1 - m_2}\left[\frac{1}{D-m_1} - \frac{1}{D-m_2}\right] f(x)$$

$$= \frac{1}{m_1-m_2}\left[\frac{1}{D-m_1} f(x) - \frac{1}{D-m_2} f(x)\right] = \frac{1}{m_1-m_2}\left[e^{m_1 x} \int e^{-m_1 x} f(x) dx - e^{m_2 x} \int e^{-m_2 x}\right.$$

A. <u>General method</u>

olve (1) $\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y=e^{e^x}$

Sol: for C.F, A.E is $D^2+3D+2=0 \Rightarrow (D+2)(D+1)=0$

Hence $D=-2,-1$ and C.F $= c_1e^{-2x}+c_2e^{-x}$

Here $P.I = y_p = \frac{1}{(D+2)(D+1)}e^{e^x} = \frac{1}{D+2}\left[\frac{1}{D+1}e^{e^x}\right]$

$= \frac{1}{D+2}\left[e^{-x}\int e^x . e^{e^x}dx\right]$ put $e^x=t$

$e^x dx = dt$

$= \frac{1}{D+2}\left[e^{-x}\int e^t dt\right]$

$= \frac{1}{D+2}\cdot e^{-x}. e^t = \frac{1}{D+2}\left[e^{-x}. e^{e^x}\right]$

$= e^{-2x}\int e^{2x}e^{-x}. e^{e^x}dx = e^{-2x}\int e^x e^{e^x}dx$

$y_p = e^{-2x}.e^{e^x}$

Hence the complete solution will be

$y = c_1e^{-x}+c_2e^{-2x}+e^{-2x}e^{e^x}.$

) solve $\frac{d^2y}{dx^2}+\frac{dy}{dx} = \frac{1}{1+e^x}$

WE have $(D^2+D)y = \frac{1}{1+e^x}$. Here $D=\frac{d}{dx}$

A.E is $D^2+D=0 \Rightarrow D=0,-1.$

$C.F = c_1e^{-x}+c_2$

$$P.I = y_p = \frac{1}{D(D+1)}\left(\frac{1}{1+e^x}\right) = \left(\frac{1}{D} - \frac{1}{D+1}\right)\frac{1}{1+e^x}$$

$$= \frac{1}{D}\left(\frac{1}{1+e^x}\right) - \frac{1}{D+1}\left(\frac{1}{1+e^x}\right) = \int \frac{1}{1+e^x}dx - e^{-x}\int \frac{e^x}{1+e^x}dx$$

$$= \int \frac{e^x dx}{e^x(1+e^x)} - e^{-x}\int \frac{e^x}{1+e^x}dx$$

put $1+e^x = t$

$e^x dx = dt$

$$= \int \frac{dt}{t(t-1)} - e^{-x}\log(1+e^x)$$

$$= \int \left[\frac{1}{t-1} - \frac{1}{t}\right]dt - e^{-x}\log(1+e^x)$$

$$= \log(t-1) - \log t - e^{-x}\log(1+e^x)$$

$$= \log(e^x) - \log(1+e^x) - e^{-x}\log(1+e^x)$$

$$= x - \log(1+e^x) - e^{-x}\log(e^x+1)$$

Hence the complete solⁿ is

$$y = c_1 + c_2 e^{-x} + x - \log(1+e^x) - e^{-x}\log(e^x+1)$$

3) solve $\frac{d^2y}{dx^2} + 9y = \sec 3x$

A.E is $D^2+9=0 \Rightarrow D = \pm 3i$

$C.F = c_1\cos 3x + c_2 \sin 3x$

$$P.I = \frac{1}{D^2+9}(\sec 3x) = \frac{1}{(D+3i)(D-3i)}(\sec 3x)$$

$$= \frac{1}{6i}\left[\frac{1}{D-3i} - \frac{1}{D+3i}\right]\sec 3x$$

$$= \frac{1}{6i}\frac{1}{D-3i}\sec 3x - \frac{1}{6i}\frac{1}{D+3i}\sec 3x$$

Now $\frac{1}{D-3i}\sec 3x = e^{3ix}\int e^{-3ix}\cdot\sec 3x\,dx$

$$= e^{3ix}\int \frac{\cos 3x - i\sin 3x}{\cos 3x}dx = e^{3ix}\int[1-i\tan 3x]$$

$$= e^{3ix}\left[x+\frac{i}{3}\log(\cos 3x)\right]$$

changing i to $-i$ we get

$$\frac{1}{D+3i}(\sec 3x) = e^{-3ix}\left[x-\frac{i}{3}\log(\cos 3x)\right]$$

Hence

$$y_p = \frac{1}{6i}\left[e^{3ix}\left\{x+\frac{i}{3}\log\cos 3x\right\} - e^{-3ix}\left\{x-\frac{i}{3}\log(\cos 3x)\right\}\right]$$

$$= \frac{x}{6i}e^{3ix} + \frac{e^{3ix}\log\cos 3x}{18} - \frac{xe^{-3ix}}{6i} + \frac{e^{-3ix}\log(\cos 3x)}{18}$$

combining like terms we get

$$= \frac{x}{3}\left[\frac{e^{3ix}-e^{-3ix}}{2i}\right] + \frac{1}{9}\left[\frac{e^{3ix}+e^{-3ix}}{2}\right]\log(\cos 3x)$$

$$= \frac{x}{3}\cdot\sin 3x + \frac{1}{9}\cos 3x\cdot\log(\cos 3x)$$

Hence the general solution will be

$$y = c_1\cos 3x + c_2\sin 3x + \frac{x}{3}\sin 3x + \frac{1}{9}\cos 3x\cdot\log(\cos 3x)$$

4.. Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$

C.F : A.E is $D^2+5D+6=0$, $\Rightarrow D=-2,-3$

C.F $= e^{-2x}+e^{-3x}$.

$$\text{P.I} = y_p = \frac{1}{(D+2)(D+3)}e^{-2x}\cdot\sec^2 x(1+2\tan x)$$

$$= \frac{1}{D+3}\cdot\left[e^{-2x}\cdot\int e^{2x}\cdot e^{-2x}\cdot\sec^2 x(1+2\tan x)\,dx\right]$$

$$= \frac{1}{D+3}\left[e^{-2x}\cdot\int \sec^2 x\cdot(1+2\tan x)\,dx\right]$$

put $\tan x = t$. $\sec^2 x\, dx = dt$

$$y_p = \frac{1}{D+3}\left[e^{-2x}\int(1+2t)\,dt\right] = \frac{1}{D+3}e^{-2x}(t+t^2)$$

$$= \frac{1}{D+3}e^{-2x}\cdot(\tan x + \tan^2 x)$$

$$= e^{-3x}\int e^{3x}\cdot e^{-2x}(\tan x + \tan^2 x)\,dx$$

$$= e^{-3x}\int e^{x}[(\tan x - 1) + \sec^2 x]\,dx$$

$$= e^{-3x}\cdot[e^{x}(\tan x - 1)] \qquad \int e^x[f(x)+f'(x)]\,dx = e^x f(x).$$

$$= e^{-2x}(\tan x - 1)$$

Hence the complete soln is

$$\boxed{y = \cancel{c_1+c_2}\; c_1 e^{-2x} + c_2 e^{-3x} + e^{-2x}(\tan x - 1)}$$

Ex 1. Solve $(D^2-1)y = (1+e^{-x})^{-2}$

A.E $D^2-1=0$, $D=\pm 1$, $C.F = c_1e^{x}+c_2e^{-x}$

$$P.I = \frac{1}{(D+1)(D-1)}[1+e^{-x}]^{-2} = \frac{1}{D+1}e^{x}\int e^{-x}(1+e^{-x})^{-2}\,dx$$

$$= \frac{1}{D+1}(-e^{x})\int(1+e^{-x})^{-2}(-e^{-x})\,dx = \frac{-1}{D+1}\left[-e^{x}(1+e^{-x})\right]^{-1}$$

$$= e^{-x}\int e^{x}\cdot e^{x}\cdot\frac{1}{1+e^{-x}}\,dx = e^{-x}\int\frac{e^{2x}\cdot(e^{x}dx)}{1+e^{x}} \qquad (1+e^{x}=t)$$

$$= e^{-x}\int\frac{(t-1)^2}{t}\,dt = e^{-x}\left[\frac{t^2}{2} - 2t + \log t\right]$$

$$= e^{-x}\left[\frac{(1+e^{x})^2}{2} - 2(1+e^{x}) + \log(1+e^{x})\right]$$

$$y = c_1e^{x}+c_2e^{-x}+e^{-x}\left[\frac{(1+e^{x})^2}{2} - 2(1+e^{x}) + \log(1+e^{x})\right]$$

B] **shortcut methods for finding P.I. in certain standard cases**

The general method disussed in the previous article will always work in the theory, it many times leads to laborious & difficult integration. To avoid this, shortcut methods of finding P.I without actual integration are developed depending upon the particular form of function $f(x)$.

Case I : **P.I. When $f(x) = e^{ax}$, a is any constant**

To find $y_p = \frac{1}{\phi(D)} e^{ax}$. We have $D e^{ax} = a e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$

$\dots \quad D^n e^{ax} = a^n e^{ax}$

$$\therefore (a_0 D^n + a_1 D^{n-1} + \dots + a_n) e^{ax} = (a_0 a^n + a_1 a^{n-1} + \dots + a_n) e^{ax}$$

$$\Rightarrow \phi(D) e^{ax} = \phi(a) e^{ax}$$

operating on both side by $\frac{1}{\phi(D)}$, We have

$$\frac{1}{\phi(D)} [\phi(D) e^{ax}] = \frac{1}{\phi(D)} (\phi(a) e^{ax}]$$

$$\text{or} \quad e^{ax} = \phi(a) \cdot \frac{1}{\phi(D)} e^{ax}$$

Dividing by $\phi(a)$ We have

$$\boxed{\frac{1}{\phi(D)} e^{ax} = \frac{1}{\phi(a)} e^{ax}} \quad \text{provided } \phi(a) \neq 0.$$

If $\phi(a) = 0$ then above rule fail and we use

$$\boxed{\frac{1}{\phi(D)} e^{ax} = x \cdot \frac{1}{\phi'(a)} e^{ax}}, \quad \phi'(a) \neq 0$$

If $\phi'(a) = 0$, then

$$\frac{1}{\phi(D)} e^{ax} = x^2 \cdot \frac{1}{\phi''(a)} e^{ax}, \quad \phi''(a) \neq 0$$

Remark 1

It can also be established that

$$\frac{1}{\cancel{\phi(D)}}\ \frac{1}{(D-a)^r\psi(D)} e^{ax} = \frac{1}{\psi(a)} \frac{x^r}{r!} e^{ax},\ \text{provided } \psi(a)\neq 0$$

Remark 2: Any constant k can be expressed as $k = k \cdot e^{0x}$

$$y_p = \frac{1}{\phi(D)} k = \frac{1}{\phi(D)} k \cdot e^{0x} = k \cdot \frac{1}{\phi(D)} e^{0x}$$

$$= k \cdot \frac{1}{\phi(0)}, \quad \phi(0)\neq 0.$$

Remark 3: If $f(x) = a^x$ then, we use $a^x = e^{x \cdot \log a}$

$$\therefore y_p = \frac{1}{\phi(D)} a^x = \frac{1}{\phi(D)} e^{x \cdot \log a} = \frac{1}{\phi(\log a)} a^x$$

Replace D with $\log a$.

If $f(x) = a^{-x}$ then we use $a^{-x} = e^{x\cdot(-\log a)}$

$$\therefore y_p = \frac{1}{\phi(D)} a^{-x} = \frac{1}{\phi(D)} e^{x\cdot(-\log a)} =$$

$$= \frac{1}{\phi(-\log a)} \cdot a^{-x}$$ Replace D with $-\log a$

Formulae for Ready References

1. $\frac{1}{D-a} e^{ax} = x \cdot e^{ax}$, 2. $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2!} e^{ax}$

3) $\frac{1}{(D-a)^3} e^{ax} = \frac{x^3}{3!} e^{ax}$ 4. $\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax}$

5. $\frac{1}{(D-a)^r\psi(D)} e^{ax} = \frac{1}{\psi(a)} \frac{1}{(D-a)^r} e^{ax} = \frac{1}{\psi(a)} \frac{x^r}{r!} e^{ax}$, $\psi(a) \neq$

Ex : Find the P.I. of followings

1) $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 6y = e^{2x}$. Ans $y = c_1 e^{6x} + c_2 e^{x} - \frac{e^{2x}}{4}$

2) $\frac{d^2y}{dx^2} - 4y = (1+e^x)^2 + 3$ Ans: $c_1 e^{2x} + c_2 e^{-2x} - 1 - \frac{2}{3}e^{x} + \frac{xe^{2x}}{4}$

3) $(D^3 - 5D^2 + 8D - 4)y = e^{2x} + 2e^{x} + 3e^{-x} + 2.$
Ans: $c_1e^{x} + (c_2x + c_3)e^{2x} + \frac{e^{2x}x^2}{2} + 2xe^{x} - \frac{e^{-x}}{6} - \frac{1}{2}$

4) $(D^2 - 5D + 6)y = 3e^{5x}$ Ans $c_1e^{2x} + c_2e^{3x} + \frac{e^{5x}}{2}$

5) $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = e^{-3x}$ P.I $= \frac{xe^{-3x}}{-2}$

6) $(D-1)^3 y = e^{x} + 2^{x} - \frac{3}{2}$ $(c_1x^2 + c_2x + c_3)e^{x} + \frac{x^3}{6}e^{x} + \frac{1}{(\log 2 - 1)^3}2^{x} + \frac{3}{2}$

7. $(D-2)^2(D+1)y = e^{2x} + 2^{-x}$. PI $= \frac{1}{3}\frac{x^2}{2!}e^{2x} + \frac{2^{-x}}{(-\log 2 - 2)^2(-\log 2 + 1)}$

Case II : P.I. when $f(x) = \sin(ax+b)$ or $\cos(ax+b)$

To find $y_p = \frac{1}{\phi(D^2)}\sin(ax+b)$ or $\frac{1}{\phi(D^2)}\cos(ax+b)$

We have

$$\frac{1}{\phi(D^2)}\sin(ax+b) = \frac{1}{\phi(-a^2)}\sin(ax+b) \quad , \text{provided } \phi(-a^2) \neq 0$$

If $\phi(-a^2) = 0$ the above rule fails and we use

$$\frac{1}{\phi(D^2)}\sin(ax+b) = x \cdot \frac{1}{\phi'(-a^2)}\sin(ax+b) \quad , \text{provided } \phi'(-a^2) \neq 0.$$

If $\phi'(-a^2) = 0$ then

$$\frac{1}{\phi(D^2)}\sin(ax+b) = x^2 \cdot \frac{1}{\phi''(-a^2)}\sin(ax+b) \quad \text{provided } \phi''(-a^2) \neq 0$$

similarly we can find formulae for $\cos(ax+b)$.

Additional Results

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax, \quad \phi(-a^2)\neq 0 \quad \text{Replace } D^2 \text{ with } -a^2$$

$$\frac{1}{\phi(D^2)}\cos ax = \frac{1}{\phi(-a^2)}\cos ax, \quad \phi(-a^2)\neq 0 \quad \text{Replace } D^2 \text{ with } -a^2$$

for case failure, it can also be established that

$$\frac{1}{D^2+a^2}\sin(ax+b) = -\frac{x}{2a}\cos(ax+b)$$

$$\frac{1}{D^2+a^2}\cos(ax+b) = \frac{x}{2a}\sin(ax+b)$$

Useful Formulae

$$\sin^2 x = \frac{1-\cos 2x}{2} = \frac{e^{0x}}{2} - \frac{\cos 2x}{2}$$

$$\cos^2 x = \frac{1+\cos 2x}{2} = \frac{e^{0x}}{2} + \frac{\cos 2x}{2}$$

$$\sin A\cdot\cos B = \frac{1}{2}[\sin(A+B) + \sin(A-B)]$$

$$\cos A\cdot\cos B = \frac{1}{2}[\cos(A+B) + \cos(A-B)]$$

$$\sin A\cdot\sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

Note $D^3 = D^2\cdot D$, $D^4 = (D^2)^2$, $D^5 = (D^2)^2 D$. Always replace D^2 by $-[a^2]$ and keep D as it is. To get D^2 in the denominator, rationalize the denominator and then replace D^2 by $-a^2$. Now numerator will contain an operator in D, therefore open the bracket

Ex solve the following.

1) $(D^2+2D+1)y = 4\sin 2x$

2) $\frac{d^3y}{dx^3} + 4\frac{dy}{dx} = \sin 2x$

3) $(D^2+1)y = \sin x \cdot \sin 2x$

4) $\frac{d^2y}{dx^2} + 4y = \cos x \cdot \cos 2x \cdot \cos 3x$

5) $(D^5 - D^4 + 2D^3 - 2D^2 + D - 1)y = \cos x$

6) $(D^4 - m^4)y = \sin mx$, $y = c_1e^{mx} + c_2e^{-mx} + c_3\cos mx + c_4\sin mx + \frac{x}{4m^3}\cos mx$

7) $(D^3+D)y = \cos x$. Ans: $y = c_1 + c_2\cos x + c_3\sin x - \frac{x\cos x}{2}$

$(D^3+4D)y = \cos 2x$

Case III : P.I. When $f(x) = \cosh(ax+b)$ or $\sinh(ax+b)$

To find $y_p = \frac{1}{\phi(D^2)}\cosh(ax+b)$ or $\frac{1}{\phi(D^2)}\sinh(ax+b)$ we use

$$\frac{1}{\phi(D^2)}\cosh(ax+b) = \frac{1}{\phi(a^2)}\cosh(ax+b) \qquad \phi(a^2) \neq 0$$

$$\frac{1}{\phi(D^2)}\sinh(ax+b) = \frac{1}{\phi(a^2)}\sinh(ax+b) \qquad \phi(a^2) \neq 0$$

[Replace D^2 by a^2]

Ex ① $\frac{d^3y}{dx^3} - 4\frac{dy}{dx} = 2\sinh 2x$.

2) $(D^2-1)y = \cos 2x$

③ $(D^4 - m^4)y = \cosh mx$

Case IV P.I. When $f(x) = x^m$

To find $y_p = \frac{1}{\phi(D)} x^m$. We write $\frac{1}{\phi(D)} x^m = [\phi(D)]^{-1} x^m$

we shall now expand $[\phi(D)]^{-1}$ in ascending powers of D as far as the term in D^m and operate on x^m term by term. Since $(m+1)^{th}$ and higher derivatives of x^m will be zero, we neednot consider term beyond D^m.

Important formulae

$$\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \cdots$$

Also note that $D^n(x^n) = n!$ & $D^{n+1}(x^n) = 0$

Note: To find $y_p = \frac{1}{\phi(D)} x^m$

1) we always take constant term common from the denominator and use the formulae $(1+x)^{-1}, (1-x)^{-1}$ $(1+x)^n, (1-x)^n$.

2) If the constant term is absent in the denominator then the minimum power of D is taken as common from the denominator.

eg. $$\frac{1}{D^2-3D-2} x^m = \frac{1}{-2\left[1 - \left(\frac{D^2-3D}{2}\right)\right]} x^m$$

$$\frac{1}{D^2-3D+3} x^m = \frac{1}{3\left[1 + \left(\frac{D^2-3D}{3}\right)\right]} x^m$$

$$\frac{1}{D^3-3D^2+2D}x^m = \frac{1}{2D\left[1+\left(\frac{D^2-3D}{2}\right)\right]}x^m$$

Ex: find the P.I of followings

1) $\frac{d^2y}{dx^2}-\frac{dy}{dx}+y = x^3-3x^2+1$ (2) $(D^3+D)y = x^5$

Soln $y_P = \frac{1}{D^2-D+1}(x^3-3x^2+1) = \frac{1}{[1-D+D^2]}(x^3-3x^2+1)$

$= [1-(D-D^2)]^{-1}(x^3-3x^2+1)$

Expanding by Binomial theorem upto D^3 terms

$= [1+(D-D^2)+(D-D^2)^2+(D-D^2)^3+\cdots][x^3-3x^2+1)$

$= [1+D-D^2+D^2-2D^3+\cdots+D^3+\cdots](x^3-3x^2+1)$

$= [1+D-D^3)[x^3-3x^2+1)$

$\boxed{y_P = x^3-6x-5}$

2) $\frac{d^3y}{dx^3}+8y = x^4+2x+1$

$y_P = \frac{1}{D^3+8}(x^4+2x+1) = \frac{1}{8}\left[\frac{1}{1+\frac{D^3}{8}}\right](x^4+2x+1)$

$= \frac{1}{8}\left(1+\frac{D^3}{8}\right)^{-1}(x^4+2x+1)$ Expanding by Binomial theorem upto D^4.

$= \frac{1}{8}\left\{1-\left(\frac{D^3}{8}\right)+\left(\frac{D^3}{8}\right)^2+\cdots\right\}(x^4+2x+1) = \left\{\frac{1}{8}\left[1-\frac{D^3}{8}\right](x^4+2x+1)\right\}$

$= \frac{1}{8}\left\{x^4+2x+1-\frac{1}{8}\cdot 24x\right\} = \frac{1}{8}\left\{x^4+2x+1-3x\right\}$

$= \frac{1}{8}\left\{x^4-x+1\right\}.$

3) $(D^4+6D^2+25)\,y = x^4+x^2+1$

Ans: $y_p = \frac{1}{25}\left[x^4 - \frac{47}{25}x^2 + \frac{589}{625}\right]$

4) $(D^3-3D^2+3D-1)\,y = 2x^3-3x^2+1.$

$y_p = -(2x^3+15x^2+54x+85)$

5) $(D^3-2D+4)\,y = 3x^2-5x+2.$

$$y_p = \frac{1}{D^3-2D+4}\left[3x^2-5x+2\right]$$

$$= \frac{1}{4}\left[1+\frac{D^3-2D}{4}\right]^{-1}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1-\left(\frac{D^3-2D}{4}\right)+\left(\frac{D^2-2D}{4}\right)^2+\cdots\right\}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1-\frac{D^3-2D}{4}+\frac{\cdots+4D^2}{16}+\cdots\right\}(3x^2-5x+2)$$

$$= \frac{1}{16}\left\{\frac{4}{4}-\frac{D^3+2D}{4}+\frac{4D^2}{4}\right\}(3x^2-5x+2)$$

$$= \frac{1}{16}\left\{3x^2-5x+2+\frac{2}{4}\{6x-5\}+\frac{4(6)}{4}\right\}$$

$$= \frac{1}{16}\left\{3x^2-5x+2+6x-\frac{5}{4}+6\right\}$$

$$= \frac{1}{16}\left\{3x^2+7x+16\right\}$$

$$y_p = \frac{1}{D^3-2D+4}(3x^2-5x+2) = \frac{1}{4\left[1+\frac{D^3-2D}{4}\right]}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1+\frac{D^3-2D}{4}\right\}^{-1}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1-\left(\frac{D^3-2D}{4}\right)+\left(\frac{D^3-2D}{4}\right)^2+\cdots\right\}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{1+\frac{2D}{4}+\frac{4D^2}{16}\right\}(3x^2-5x+2)$$

$$= \frac{1}{4}\left\{(3x^2-5x+2)+\frac{1}{2}(6x-5)+\frac{1}{4}\cdot(6)\right\}$$

$$= \frac{1}{4}\left\{3x^2-5x+2+3x-\frac{5}{2}+\frac{3}{2}\right\} = \frac{1}{4}\left[3x^2-2x+1\right]$$

Case V : P.I. When $f(x) = e^{ax}\cdot v$. Where v is any function of x

To find $y_p = \frac{1}{\phi(D)}(e^{ax}\cdot v)$ we have

$$\boxed{\frac{1}{\phi(D)}(e^{ax}\cdot v) = e^{ax}\cdot\frac{1}{\phi(D+a)}v}$$

① Find P.I. of followings

① $(D^2-4D+3)y = x^3e^{2x}$

$$y_p = \frac{1}{D^2-4D+3}(x^3e^{2x}) = e^{2x}\cdot\frac{1}{(D+2)^2-4(D+2)+3}x^3$$

$$= e^{2x}\frac{1}{D^2-1}x^3 = -e^{2x}[1-D^2]^{-1}x^3$$

$$= -e^{2x}\{1+D^2+D^4+\cdots] = -e^{2x}[x^3+6x]$$

Solve $(D^3 - D^2 + 3D + 5)y = e^x \cdot \cos 3x$

$$C.F = c_1 e^{-x} + e^x[c_2 \cos 2x + c_3 \sin 2x]$$

$$y_p = \frac{1}{D^3 - D^2 + 3D + 5}(e^x \cdot \cos 3x)$$

$$= e^x \frac{1}{(D+1)^3 - (D+1)^2 + 3(D+1) + 5} \cos 3x$$

$$= e^x \frac{1}{D^3 + 3D^2 + 3D + 1 - D^2 - 2D - 1 + 3D + 3 + 5} \cos 3x$$

$$= e^x \frac{1}{D^3 + 2D^2 + \cancel{6D}\, 4D + 8} \cos 3x$$

$$= e^x \frac{1}{D \cdot D^2 + 2D^2 + 4D + 8} \cos 3x$$

$$= e^x \frac{1}{-9D + 2(-9) + 4D + 8} \cos 3x = e^x \frac{1}{4D - 9D + 8 - 18} \cos 3x$$

$$= e^x \frac{1}{-5D - 10} \cos 3x = -e^x \frac{1}{5(D+2)} \cos 3x$$

$$= -e^x \frac{1}{5} \frac{D-2}{(D^2 - 4)} \cos 3x$$

$$= -\frac{e^x}{5} \frac{D-2}{-9-4} \cos 3x = -\frac{e^x}{5} \cdot \frac{1}{-13} (D-2) \cos 3x$$

$$= \frac{e^x}{65}[-3\sin 3x - 2\cos 3x] = -\frac{e^x}{65}[3\sin 3x + 2\cos 3x]$$

3) $(D^2-4D+4)y = e^{2x}\cdot\sin 3x$

<u>Ans:</u> $y = (c_1+c_2x)e^{2x} - \frac{1}{9}e^{2x}\cdot\sin 3x$.

4) $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1+x^2)$

Ans: $y = c_1e^{-x} + c_2e^{-2x} + c_3e^{3x} - \frac{e^{2x}}{12}\left[\frac{169}{72} + x^2 + \frac{5x}{6}\right]$

Case VI: P.I. When $f(x) = x^m\cdot\sin ax$ (or) $x^m\cos ax$.

To find $y_p = \frac{1}{\phi(D)}x^m\sin ax$ or $\frac{1}{\phi(D)}x^m\cos ax$ We have

$$\frac{1}{\phi(D)}x^m[\cos ax + i\sin ax] = \frac{1}{\phi(D)}x^m e^{iax}$$

$$= e^{iax}\frac{1}{\phi(D+ia)}x^m$$

Now $\frac{1}{\phi(D+ia)}x^m$, can be evaluated by method of case IV and equating real & imaginary parts, we get the required results.

Ex: solve $(D^4+2D^2+1)y = x^2\cos x$

A.E $(D^2+1)^2 = 0 \Rightarrow D = \pm i, \pm i$

$y_c = (c_1x+c_2)\cos x + (c_3x+c_4)\sin x$

Ex ① $(D^4+18D^2+81)y = x^2\sin 3x$

for P.I. We have

$$\frac{1}{(D^2+1)^2}[x^2(\cos x + i\sin x)] = \frac{1}{(D^2+1)^2}x^2e^{ix}$$

$$= e^{ix}\frac{1}{[(D+i)^2+1]^2}x^2 = e^{ix}\frac{1}{(D^2+2iD)^2}x^2$$

$$= e^{ix} \frac{1}{-4D^2\left(1-\frac{iD}{2}\right)^2} x^2 = -\frac{e^{ix}}{4} \frac{1}{D^2}\left(1-\frac{iD}{2}\right)^{-2} x^2$$

$$= -\frac{e^{ix}}{4} \frac{1}{D^2}\left[1 + iD - \frac{3}{4}D^2 + \cdots\right] x^2$$

$$= -\frac{e^{ix}}{4} \frac{1}{D^2}\left[x^2 + 2ix - \frac{3}{2}\right]$$

$$= -\frac{e^{ix}}{4}\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right] \quad \text{integrating twice}$$

$$= -\frac{1}{4}\left[\cos x + i\sin x\right]\left[\frac{x^4}{12} + \frac{ix^3}{3} - \frac{3}{4}x^2\right]$$

Equating real & imaginary parts of both side

$$\frac{1}{(D^2+1)^2} x^2\cos x = -\frac{1}{4}\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x + \frac{1}{12}x^3\sin x$$

Hence

$$y = (c_1x + c_2)\cos x + (c_3x + c_4)\sin x + \frac{x^3\sin x}{12} - \frac{x^4 - 9x^2}{48}\cos x$$

solve ~~$(D^2 - 3D + 2)y = x^2 \sin x$~~

$$\frac{1}{D^2-3D+2} x^2 \cancel{[\cos x + i\sin x]} = \frac{1}{D^2-3D+2} x^2 e^{ix}$$

$$\cancel{= e^{ix}} \frac{1}{(D+i)^2 - 3(D+i) + 2} x^2 = e^{ix} \frac{1}{D^2 - 2iD - 1 - 3D - 3i + 2} x^2$$

$$= e^{ix} \frac{1}{D^2}$$

solve $(D^2+1)y = x^2 \cdot \sin x$

$D^2+1=0, \Rightarrow D=\pm i \quad y_c = c_1\cos x + c_2 \sin x$

For $y_p = \dfrac{1}{D^2+1}(x^2 \cdot \sin x)$

consider

$$\frac{1}{D^2+1}x^2[\cos x + i\sin x] = \frac{1}{D^2+1}x^2 e^{ix} = e^{ix}\frac{1}{(D+i)^2+1}x^2$$

$$= e^{ix}\frac{1}{D^2+2iD}x^2 = e^{ix}\frac{1}{2iD\left[\frac{D^2}{2iD}+1\right]}x^2$$

$$= e^{ix}\frac{1}{2iD}\left[1-\frac{iD}{2}\right]^{-1}x^2 = e^{ix}\frac{1}{2iD}\left[1+\frac{iD}{2}-\frac{D^2}{4}\right]x^2$$

$$= e^{ix}\frac{1}{2iD}\left[x^2 - ix - \frac{1}{2}\right] = \cancel{(\cos x + i\sin x)(x^2-1)}$$

$$= [\cos x + i\sin x]\frac{1}{2i}\left[\frac{x^3}{3} - \frac{ix^2}{2} - x\right] = [\cos x + i\sin x]\left[\left(\frac{x^3}{6}-\frac{x}{2}\right)\frac{1}{i} + \frac{x^2}{4}\right]$$

$$= (\cos x + i\sin x)\left[\frac{x^2}{4} - \left(\frac{x^3}{6}-\frac{x}{2}\right)i\right]$$

Equaling imaginary part on both side we get

$$\frac{1}{D^2+1}(x^2\sin x) = \frac{x^2}{4}\sin x - \left(\frac{x^3}{6}-\frac{x}{2}\right)\cos x.$$

Ex solve $(D^2+1)y = x\sin x$

Find P.I. $(D^2+3D+2)y = x^2\sin x$

Case XII P.I. When $f(x) = xV$. V being any function of x

To find $\frac{1}{\phi(D)}(xV)$, We have

$$y_p = \frac{1}{\phi(D)}[xV] = \left[x - \frac{1}{\phi(D)}\phi'(D)\right]\frac{1}{\phi(D)}V$$

1. The rule xV is applied if
   (i) Power of x is one
   (ii) $\frac{1}{\phi(D)}V$ is not a case of failure.
2. If power of x is one and $\frac{1}{\phi(D)}V$ is a case of failure then do not apply xV rule. In this case apply rule given by case (VI)

eg. $y_p = \frac{1}{D^2+1}(x \cdot \sin x)$

Here $\frac{1}{D^2+1}\sin x$ is a case of failure. Therefore use case (VI) method,

Solve ① $\frac{d^2y}{dx^2} + 4y = x\sin x$ ② $(D^2+4)y = x\sin^2 x$

A.E is $D^2+4=0$, $D = \pm 2i$

$C.F = c_1\cos 2x + c_2\sin 2x$

$$P.I = \frac{1}{D^2+4}(x\sin x) = \left[x - \frac{1}{D^2+4}\cdot 2D\right]\frac{1}{D^2+4}\sin x$$

$$= \left[x - \frac{1}{D^2+4}\cdot 2D\right]\frac{1}{-1+4}\sin x = \left[x - \frac{2D}{D^2+4}\cdot\right]\frac{1}{3}\sin x$$

$$= \frac{1}{3}\left[x\cdot\sin x - \frac{2D}{D^2+4}\sin x\right] = \frac{1}{3}\left[x\cdot\sin x - \frac{2D}{3}\sin x\right]$$

$$= \frac{1}{3}\left[x\sin x - \frac{2\cos x}{3}\right]$$

Hence

$$y = c_1\cos 2x + c_2\sin 2x + \frac{x\sin x}{3} - \frac{2}{9}\cos x$$

2) $(D^2-2D+1)y = xe^x\sin x$

A.E. $D^2-2D+1=0$. $D = 1, 1$. $CF = (c_1x+c_2)e^x$

$$P.I = \frac{1}{(D-1)^2}(xe^x\sin x) = e^x\frac{1}{(D+1-1)^2}(x\sin x)$$

$$= e^x\left[x - \frac{2D}{D^2}\cdot\right]\frac{1}{D^2}\sin x = e^x\left[x-\frac{2}{D}\right]\frac{1}{-1}\sin x$$

$$= -e^x[x\cdot\sin x - 2(-\cos x)] = -e^x[x\sin x + 2\cos x]$$

$$y = (c_1x+c_2)e^x - e^x[x\sin x + 2\cos x]$$

Exercise

solve ① $(D^2+2D+1)y = 2\cos x + 3x + 2 + 3e^x$

$$y = (c_1x+c_2)e^{-x} + \frac{3e^x}{4} + \sin x + 3x - 4$$

2) $(D^3-1)y = (1+e^x)^2$ $y=$

$$y = c_1e^x + c_2e^{-\frac{1}{2}x}\left[c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right] - 1 + \frac{2}{3}xe^x + \frac{1}{7}e^{2x}$$

3) $(D-1)^2(D^2+1)^2y = \sin^2\frac{x}{2}$

$$y = (c_1x+c_2)e^x + (c_3x+c_4)\cos x + (c_5x+c_6)\sin x$$

4) $(D^4-2D^3-3D^2+4D+4)y = x^2e^x$

$$y = (c_1x+c_2)e^{2x} + (c_3x+c_4)e^{-x} + \frac{e^x}{4}\left[x^2+2x+\frac{7}{2}\right]$$

$(D^4+1)y = 2\sinh x \cdot \sin x$

$y = e^{\frac{x}{\sqrt{2}}}\left[c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right] + e^{-\frac{x}{\sqrt{2}}}\left[c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right]$

$- \frac{2}{3}\sin x \cdot \sinh x.$

5. $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = e^{2x}(1+x)$

$y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{3x} - \frac{e^{2x}}{12}\left(x + \frac{17}{12}\right)$

6) $(D^2-1)y = x\cdot\sin x + (1+x^2)e^x$

$y = c_1 e^{x} + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{12}(2x^3 - 3x^2 + 9x)$

7) $(D^2-4D+4)y = e^x \cdot \cos^2 x$

$y = (c_1 x + c_2)e^{2x} + \frac{e^x}{2}\left[1 - \frac{1}{25}(4\sin 2x + 3\cos 2x)\right]$

8). $(D^2+D+1)y = x\cdot\sin x$

$y = e^{-\frac{1}{2}x}\cdot\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] - x\cos x + 2\cos x + \sin x$

9) $(D^2+2D+1)y = x e^{-x}\cdot\cos x$

$y = (c_1 x + c_2)e^{-x} + e^{-x}(-x\cos x + 2\sin x)$

10. $(D^2+1)y = x^2 \sin 2x$

$y = c_1\cos x + c_2 \sin x - \frac{1}{3}\left(x^2 - \frac{26}{9}\right)\sin 2x - \frac{8}{9}x\cos 2x$

11. $(D^2+4)y = x\cdot\sin^2 x$

$y = c_1\cos 2x + c_2\sin 2x + \frac{x}{8} - \frac{1}{32}(x\cos 2x + 2x^2\sin 2x).$

method of variation of parameters

When we have to solve eqn of the type $a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$ where a, b, c are constants and X, any function of x, we also have an alternative method of variation of parameters

Let $C.F = Ay_1 + By_2$ then

$P.I = uy_1 + vy_2$

Where $u = \int \frac{-y_2 X}{W} dx$, $v = \int \frac{y_1 X}{W} dx$

Where $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$

solve the following differential equations by method of variations of parameters

1) $\frac{d^2y}{dx^2} + 4y = \tan 2x.$

$A.E = D^2 + 4 = 0, \quad D = \pm 2i$

$C.F = c_1 \cos 2x + c_2 \sin 2x$

$= c_1 y_1 + c_2 y_2$

$y_1 = \cos 2x \qquad y_2 = \sin 2x$

$y_1' = -2\sin 2x \qquad y_2' = 2\cos 2x$

$P.I = uy_1 + vy_2$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2.$

$u = \int \frac{-y_2 X}{W} dx = \int \frac{-\sin 2x \cdot \tan 2x}{2} = \frac{1}{2}\int \frac{\sin^2 2x}{\cos 2x} dx$

$$= -\frac{1}{2}\int\left[\frac{1-\cos^2 2x}{\cos 2x}\right]dx = -\frac{1}{2}\left[\int \sec 2x\,dx - \int \cos 2x\,dx\right]$$

$$= -\frac{1}{2}\left[\frac{\log(\sec 2x+\tan 2x)}{2} - \frac{\sin 2x}{2}\right]$$

$$= -\frac{1}{4}\log[\sec 2x+\tan 2x] + \frac{1}{4}\sin 2x.$$

$$v_1 = \int \frac{y_1 X}{W}dx = \int \frac{\cos 2x\cdot\tan 2x}{2}dx = \int\frac{\sin 2x}{2}dx$$

$$= -\frac{\cos 2x}{4}$$

$\therefore$ P.I $= uy_1 + vy_2$

$$= \left[-\frac{1}{4}\log[\sec 2x+\tan 2x] + \frac{1}{4}\sin 2x\right]\cos 2x + \left(-\frac{\cos 2x}{4}\right)\sin 2x$$

$$= -\cos 2x\cdot\frac{1}{4}[\log(\sec 2x+\tan 2x)]$$

$$y = c_1\cos 2x + c_2\sin 2x - \cos 2x\cdot\frac{1}{4}\log(\sec 2x+\tan 2x)$$

2) $(D^2-6D+9)y = \frac{e^{3x}}{x^2}$

$$y = (c_1x+c_2)e^{3x} - e^{3x}(1+\log x)$$

3) $(D^2-2D+2)y = e^x\tan x$

$$y = e^x(c_1\cos x + c_2\sin x) - e^x\cos x\cdot\log(\sec x+\tan x)$$

4) $(D^2+4)y = \sec 2x$

$$y = A\cos x + B\sin x - x\cos x + \sin x\cdot\log(\sin x)$$

5) $(D^3+D)y = \operatorname{cosec} x.$

$$y = A + B\cos x + C\sin x + \log[\operatorname{cosec} x - \cot x] - \cos x\cdot\log\sin x - x\sin x.$$

6) $\left(\frac{d^2y}{dx^2}+4y\right) = x\cdot\sin x$. $\quad y = A\cos x + B\sin x + \frac{x}{2}\sin x - \frac{x^2}{4}\cos x.$

7) $(D^2+3D+2)y = \sin e^x$. $\quad y = c_1e^{-x} + c_2e^{-2x} - e^{-2x}\sin e^x$

Equations reduciable to linear with constant coefficients

we shall now study two types of linear differential equations with variable coefficients which can be reduced to the case of linear differential equations with constant coefficients by suitable transformations of variables.

Cauchy's OR Euler's Homogeneous Linear Differential Eqn's

An equation of the type

$$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + \cdots + a_{n-1}xD + a_n)y = F(x)$$

where $a_0, a_1, a_2 \ldots a_n$ are constants, is called cauchy's Homogeneous eqn. sometimes attributed as Euler also. It may also be written as

$$a_0x^n\frac{d^ny}{dx^n} + a_1x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2}\frac{d^{n-2}y}{dx^{n-2}} + \cdots + a_{n-1}x\frac{dy}{dx} + a_ny = F(x)$$

It can be reduced to linear differential equation with constant coefficients by putting $x = e^z$ or $z = \log x$

Now $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\cdot\frac{dy}{dz}$ Here we took

or $x\frac{dy}{dx} = \frac{dy}{dz} = Dy$ Where $D = \frac{d}{dz}$

Also $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\left[\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}\right]$

$$= -\frac{1}{x^2}\frac{dy}{dx} + \frac{1}{x^2}\left(\frac{d^2y}{dx^2}\right)\frac{1}{x} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

Hence

$$x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz} = D^2y - Dy = D(D-1)y$$

Similarly we can show that

$$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \quad \text{and so on.}$$

$$\vdots$$

$$x^r\frac{d^ry}{dx^r} = D(D-1)(D-2)\cdots(D-r+1)y$$

Making these substitutions in (1), it can be reduced to linear differential equations with constant coefficients.

Ex: Solve the following differential equations with variable coefficients.

1) $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^5$ — (1)

put $z = \log x$ or $x = e^z$ and $D = \frac{d}{dz}$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$, $x\frac{dy}{dx} = Dy$ then eq^n (1) transformed into

$$D(D-1)y - 4Dy + 6y = e^{5z} \qquad D = \frac{d}{dz}$$

ie $(D^2 - D - 4D + 6)y = e^{5z}$

$(D^2 - 5D + 6)y = e^{5z}$ — (2)

which is linear diff. eq^n with constant coefficients in y & z. Now.

A.E is $D^2 - 5D + 6 = 0.$ $\Rightarrow$ $D = 2, 3$

$$y_c = c_1 e^{2z} + c_2 e^{3z}.$$

and $y_p = \dfrac{1}{D^2-5D+6} e^{5z} = \dfrac{1}{(5)^2-5(5)+6} e^{5z} = \frac{1}{6} e^{5z}$

$\therefore$ The general solution in terms of y & z is

$$y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{6} e^{5z}.$$

changing to y & x we have

$$y = c_1 x^2 + c_2 x^3 + \frac{1}{6} x^5$$

2) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 4y = \cos(\log x) + x\sin(\log x)$

$$y = e^z[A\cos\sqrt{3}z + B\sin\sqrt{3}z] + \frac{1}{13}[3\cos z - 2\sin z] + \frac{1}{2}e^z \sin z$$

$$= x[A\cos\sqrt{3}\log x + B\sin\sqrt{3}\log x] + \frac{1}{13}[3\cos\log x - 2\sin(\log x)] + \frac{1}{2}x \cdot \sin(\log x)$$

3) $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$

<u>Ans</u>: $y = \dfrac{c_1}{x} + x[c_2\cos(\log x) + c_3 \sin \log x] + 5x + \dfrac{2}{x}\log x$

4) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 5y = x^2 \sin(\log x).$

Ans: $y = x^2[c_1\cos(\log x) + c_2 \sin(\log x)] - \frac{1}{2}x^2(\log x)\cdot\cos(\log x).$

5) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^2 + 2\log x$

Ans: $y = c_1 x^4 + \dfrac{c_2}{x} - \dfrac{x^2}{6} - \dfrac{1}{2}\log x + \dfrac{3}{8}$

6) $x^2 \dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + \dfrac{y}{x} = \log x$

$$y = \frac{c_1}{x} + \sqrt{x}\left[c_2\cos\frac{\sqrt{3}}{2}\log x + c_3 \sin\frac{\sqrt{3}}{2}\log x\right] + \frac{x}{2}\left(\log x - \frac{3}{2}\right)$$

Legendre's linear Equation

An equation of the type

$$a_0(ax+b)^n \frac{d^ny}{dx^n} + a_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots\dots + a_n y = F(x)$$

Where $a_0, a_1, a_2 \cdots a_n$ are constants is called Legendres Linear equations.

In case of such equations we put $ax+b = e^z$, $z = \log(ax+b)$ to reduce it to linear with constant coefficients.

If we put $ax+b = z$ or $z = \log(ax+b)$ then

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{a}{ax+b}\cdot\frac{dy}{dz}$$

$$\Rightarrow (ax+b)\frac{dy}{dx} = a\cdot\frac{dy}{dz} = a\cdot Dy \quad \because D = \frac{d}{dz}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{a}{ax+b}\cdot\frac{dy}{dz}\right) = \frac{-a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a}{ax+b}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= \frac{-a^2}{(ax+b)^2}\frac{dy}{dz} + \frac{a^2}{(ax+b)^2}\frac{d^2y}{dz^2}$$

$$= \frac{a^2}{(ax+b)^2}\left[\frac{d^2y}{dz^2} - \frac{dy}{dz}\right]$$

$$\Rightarrow (ax+b)^2\frac{d^2y}{dx^2} = a^2[D^2-D]y = a^2D(D-1)y.$$

Similarly we shall get

$$(ax+b)^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y \quad \text{and so on.}$$

If we make these substitutions in linear differential equation (Legendre's) we shall see that it has been transformed into one with constant coefficients.

Ex : solve the following differential equations

① $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x.$

solⁿ: Given equation is Legendre's linear differential eqⁿ.

put $z = \log(2x+3)$ or $2x+3 = e^z$

and let $D = \frac{d}{dz}$,

$(2x+3)^2\frac{d^2y}{dx^2} = 2^2 \cdot D(D-1)y = 4D(D-1)y$

$(2x+3)\frac{dy}{dx} = 2 \cdot Dy$ We have

$4 \cdot D(D-1)y - 2 \cdot 2Dy - 12y = 6 \cdot \frac{e^z-3}{2} = 3 \cdot (e^z-3)$

ie $[4D^2 - 4D - 4D - 12]y = 3(e^z-3)$

$(D^2-2D-3)y = \frac{3}{4}(e^z-3)$

A.E is $D^2-2D-3 = 0$ $D = 3, -1$

$C.F = c_1 e^{3z} + c_2 e^{-z}$

$P.I = \frac{1}{D^2-2D-3} \cdot \frac{3}{4}(e^z-3) = \frac{3}{4}\frac{1}{D^2-2D-3}e^z - \frac{9}{4}\frac{1}{D^2-2D-3}e^{0x}$

$= \frac{3}{4}\frac{1}{1-2-3}e^z - \frac{9}{4}\cdot\frac{1}{-3} = \frac{3}{-16}e^z + \frac{3}{4}$

$\therefore y = c_1 e^{3z} + c_2 e^{-z} - \frac{3}{16}e^z + \frac{3}{4}$

$= c_1(2x+3)^3 + c_2(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$

Ex: solve the following differential equations

① $(2x+3)^2 \frac{d^2y}{dx^2} - 2(2x+3)\frac{dy}{dx} - 12y = 6x.$

solⁿ: Given equation is Legendre's linear differential eqⁿ.

put $z = \log(2x+3)$ or $2x+3 = e^z$

and let $D = \frac{d}{dz}$,

$(2x+3)^2\frac{d^2y}{dx^2} = 2^2 \cdot D(D-1)y = 4D(D-1)y$

$(2x+3)\frac{dy}{dx} = 2 \cdot Dy$ We have

$4 \cdot D(D-1)y - 2 \cdot 2Dy - 12y = 6 \cdot \frac{e^z-3}{2} = 3 \cdot (e^z-3)$

ie $[4D^2 - 4D - 4D - 12]y = 3(e^z-3)$

$(D^2 - 2D - 3)y = \frac{3}{4}(e^z-3)$

A.E is $D^2 - 2D - 3 = 0$ $D = 3, -1$

$C.F = c_1 e^{3z} + c_2 e^{-z}$

$P.I = \frac{1}{D^2-2D-3} \cdot \frac{3}{4}(e^z-3) = \frac{3}{4}\frac{1}{D^2-2D-3}e^z - \frac{9}{4}\frac{1}{D^2-2D-3}e^{0x}$

$= \frac{3}{4}\frac{1}{1-2-3}e^z - \frac{9}{4} \cdot \frac{1}{-3} = \frac{3}{-16}e^z + \frac{3}{4}$

$\therefore y = c_1 e^{3z} + c_2 e^{-z} - \frac{3}{16}e^z + \frac{3}{4}$

$= c_1(2x+3)^2 + c_2(2x+3)^{-1} - \frac{3}{16}(2x+3) + \frac{3}{4}$

Solve 2: $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

Sol^n Given eq^n is Legendres linear diff eq^n

put $z = \log(3x+2)$ or $(3x+2) = e^z$ and let $D = \frac{d}{dz}$

and $$(3x+2)\frac{dy}{dx} = 3Dy$$

$$(3x+2)^2 \frac{d^2y}{dx^2} = 3^2 \cdot D(D-1)y$$

we have

$$3^2 \cdot D(D-1)y + 3\cdot 3Dy - 36y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1$$

ie $$(D^2-4)y = \frac{1}{27}\left[e^{2z}-1\right]$$

A.E is $D^2-4=0$ $D = \pm 2$

$$C.F = c_1 e^{2z} + c_2 e^{-2z}$$

$$P.I = \frac{1}{D^2-4}\frac{1}{27}(e^{2z}-1) = \frac{1}{27}\frac{1}{D^2-4}(e^{2z}-1)$$

$$= \frac{1}{27}\left[\frac{1}{D^2-4}e^{2z} - \frac{1}{D^2-4}e^{0z}\right] = \frac{1}{108}\left[ze^{2z}+1\right]$$

Hence the general solution in y & z is

$$y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}\left[ze^{2z}+1\right]$$

The general solution in y & x is

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2 \cdot \log(3x+2)+1\right]$$

Ex 3. solve $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 2\sin[\log(1+x)]$

Ans: $y = A\cos[\log(1+x)] + B\sin[\log(1+x)]$
$\qquad - \log(1+x)\cos[\log(1+x)]$.

Ex 4. $7(2+x)^2 \dfrac{d^2y}{dx^2} + 8(2+x)\dfrac{dy}{dx} + y = 4\cdot\cos[\log(2+x)]$

Ex 5. $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cdot\cos[\log(1+x)]$

$y = c_1\cos[\log(x+1)] + c_2\sin[\log(1+x)]$

Ex 6. $(2x+1)^2 \dfrac{d^2y}{dx^2} - 6(2x+1)\dfrac{dy}{dx} + 16y = 8(2x+1)^2$

$y = [c_1 + c_2\log(2x+1)](2x+1)^2 + (2x+1)^2[\log(2x+1)]^2$.

Simultaneous Linear Differential Equations

Some times in applications we come across equations, containing one independent but two or more dependent variables. For example

$$\frac{dx}{dt}+3\frac{dy}{dt}+y=t, \quad \frac{dy}{dt}-x-y=t^2$$

Here t is single independent and x and y are the two dependent variables. Such equations are called simultaneous linear differential equations. The number of equations is same as number of dependent variables.

Method of solution

Method of solution is analogous to that of solving two linear simultaneous equations in algebra either by elimination or by substitution. The equation of the system are so obtained as to get simple equation containing only one of the dependent variables and its derivatives. Then by integration, a relation between this dependent and the independent variable is found. Then either in similar way or by substitution, a relation between the second dependent and the independent variable can be easily obtained.

Ex: Solve the following simultaneous equations

1) $\frac{dx}{dt}+y=e^t, \quad \frac{dy}{dt}+x=e^{-t}$.

Sol: Writing in terms of operator $D=\frac{d}{dt}$ We have

$$Dx+y=e^t \text{ —(1) } \& \quad Dy+x=e^{-t} \text{ —(2)}$$

Solving (1) & (2) for x,

operating (1) by D, we have

$$D^2x+Dy=De^t$$
$$=e^t \quad \text{—(3)}$$

eq^n (3) − eq^n (2) We have

$$D^2x+Dy=e^t$$
$$-Dy \pm x = -e^{-t}$$
$$D^2x-x=e^t-e^{-t} \Rightarrow (D^2-1)x=e^t-e^{-t} \quad \text{—(4)}$$

This is linear differential equation with constant coefficients

A.E: $D^2-1=0 \Rightarrow D=\pm 1$

$x_c = C.F = c_1e^t+c_2e^{-t}$

$$P.I = \frac{1}{D^2-1}(e^t-e^{-t}) = \frac{1}{D^2-1}e^t-\frac{1}{D^2-1}e^{-t}, \text{ case fail}$$

$$= t\cdot\frac{1}{2D}e^t - t\frac{1}{2D}e^{-t} = t\cdot\frac{e^t}{2(1)} - t\frac{e^{-t}}{2(-1)}$$

$$= t\left[\frac{e^t+e^{-t}}{2}\right]$$

$$= t\cdot\cosh t$$

$x = x_c+x_p$

$$\boxed{x = c_1e^t+c_2e^{-t}+t\cdot\cosh t}$$

To find next solution for y: operating (2) by D we have

$$D^2y+Dx=-e^{-t} \quad \text{—(5)}$$

eq^n (5) − eq^n (1)

$$D^2y+Dx=-e^{-t}$$
$$-Dx \pm y = e^t$$
$$D^2y-y=-e^{-t}-e^t \quad \text{ie } (D^2-1)y=-e^{-t}-e^t$$

A.E is $D^2-1=0 \Rightarrow D=1,-1$

$y_c = c_1e^t+c_2e^{-t}$.

$$y_p = \frac{1}{D^2-1}(-e^{-t}-e^{t}) = -\left[\frac{1}{D^2-1}e^{-t} + \frac{1}{D^2-1}e^{t}\right]$$

$$= -\left[t\frac{1}{2D}e^{-t} + \frac{t}{2D}e^{t}\right] = -\left[t\cdot\frac{1}{-2}e^{-t} + t\cdot\frac{1}{2}e^{t}\right]$$

$$= -\frac{t}{2}\left[\frac{e^{t}-e^{-t}}{2}\right] = -t\cdot\sinh t$$

$$\boxed{y = y_c + y_p = c_1e^{t} + c_2e^{-t} - t\sinh t}$$

Solve 2. $\frac{dx}{dt} + 2x - 3y = t, \quad \frac{dy}{dt} - 3x + 2y = e^{2t}$

Writing in terms of operator $D = \frac{d}{dt}$, We have

$Dx + 2x - 3y = t$ or $(D+2)x - 3y = t$ —①

$Dy - 3x + 2y = e^{2t}$ or $(D+2)y - 3x = e^{2t}$ —②

solving for x (eliminating for y)

~~solving~~ (1) operating (1) by $(D+2)$ We have

$$(D+2)^2x - 3(D+2)y = (D+2)t$$

$$(D+2)^2x - 3(D+2)y = 1+2t \quad \text{--(3)}$$

multiply (2) by 3, We have

$$3(D+2)y - 9x = 3e^{2t} \quad \ldots(4)$$

Adding (3) & (4) we get

$$(D^2+4D-5)x = 1 + 2t + 3e^{2t}$$

This is linear differential equation with constant coefficients

A.E : $D^2+4D-5 = 0$ gives $D = -5, 1$

C.F $= c_1e^{-5t} + c_2e^{t}$

$$P.I = \frac{1}{D^2+4D-5}(1+2t) + \frac{3}{D^2+4D-5}e^{2t}$$

$$= -\frac{1}{5}\left[1 - \frac{4D+D^2}{5}\right]^{-1}(1+2t) + \frac{3e^{2t}}{4+8-5}$$

$$= -\frac{1}{5}\left[1+\frac{4D}{5}\right](1+2t) + \frac{3}{7}e^{2t} = -\frac{1}{5}\left[\frac{13}{5}+2t\right] + \frac{3e^{2t}}{7}$$

Hence the general solution for x is

$$x = c_1e^{-5t} + c_2e^{t} - \frac{13}{25} - \frac{2t}{5} + \frac{3e^{2t}}{7} \quad -(6)$$

Next the general soln for y: Diff (6) w.r.t. t

$$\frac{dx}{dt} = -5c_1e^{-5t} + c_2e^{t} - \frac{2}{7} + \frac{6}{7}e^{2t}.$$

putting the value of x & $\frac{dx}{dt}$ in eqn (1), We have

$$y = \frac{1}{3}\left[\frac{dx}{dt} + 2x - t\right]$$

$$= \frac{1}{3}\left[-5c_1e^{-5t} + c_2e^{t} - \frac{2}{5} + \frac{6}{7}e^{2t} + 2c_1e^{-5t} + 2c_2e^{t} - \frac{26}{25} - \frac{4t}{5} + \frac{6e^{2t}}{7} - t\right]$$

simplifying, We get

$$y = -c_1e^{-5t} + c_2e^{t} - \frac{12}{25} - \frac{3t}{5} + \frac{4e^{2t}}{7} \quad \text{—(7)}$$

Hence equation (6) & (7) together constitute the general solution.

Q3: solve the simultaneous linear differential eqn with given conditions

$$\frac{du}{dx} + v = \sin x, \quad \frac{dv}{dx} + u = \cos x$$

Given that when $x = 0$, then $u = 1$ & $v = 0$.

Ans : $u = \cosh x \quad v = \sin x - \sinh x$.

Q 4) $\frac{dx}{dt} + 5x - 2y = t, \quad \frac{dy}{dt} + 2x + y = 0$ having been given that $x = y = 0$ at $t = 0$. Ans: $x = -\frac{1}{27}(1+6t)e^{-3t} + \frac{1}{27}(1+3t)$

$y = -\frac{2}{27}(2+3t)e^{-3t} + \frac{2}{27}(2-3t)$

Symmetrical simultaneous Differential Equations

Def^n: Equation of type $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$, where P, Q and R are the functions of x, y and z are said to be symmetrical simultaneous differential equations.

There are mainly two methods of solving such equations. The solutions of such system consist of two independent relations of the type

$$F_1(x, y, z) = c_1 \quad \& \quad F_2(x\ y\ z) = c_2$$

A] Method of combination or grouping

If we can observe that z is missing from first group $\frac{dx}{P} = \frac{dy}{Q}$ or, may be cancelled from this equation, then it becomes a differential equation in x & y only. Solution of this will give one relation in the solution of simultaneous equation. Then we consider the second group $\frac{dy}{Q} = \frac{dz}{R}$. If it doesnot contain x, it is most ideal otherwise we cancel x (if possible) and if not try to eliminate x by the help of first relation just reached. It will then be differential equation in y & z only and after integration yields the second relation in the solution of system of simultaneous equations. Following examples will illustrate this system.

Ex 1 solve $\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{4xy^2-2z}$

consider first two terms together

$$\frac{dx}{2x} = \frac{dy}{-y} \quad \text{or} \quad \frac{dx}{x} = -2\frac{dy}{y}$$

on integrating $\log x + 2\log y = \log c_1 \Rightarrow \boxed{xy^2 = c_1}$ — (1)

Next consider first and last term together

$$\frac{dx}{2x} = \frac{dz}{4xy^2-2z},$$

using the solution (1), we remove y from this eqn & obtain

$$\frac{dx}{2x} = \frac{dz}{4c_1-2z} \Rightarrow \frac{dx}{x} = \frac{dz}{2c_1-z}$$

on integrating $\log x + \log(2c_1 - z) = \log c_2$

$$x(2c_1 - z) = c_2$$

putting back the expression for $c_1 = xy^2$, We have

$$\boxed{x(2xy^2 - z) = c_2} \quad \text{— (2)}$$

Hence (1) & (2) constitute the solution of given symmetrical equation

Ex 2. solve $\frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{x(z-2y)}$

consider first two terms $\frac{dx}{y^2} = \frac{dy}{-xy}$

or $x\,dx + y\,dy = 0$

on integrating $\boxed{x^2 + y^2 = c_1}$ — (1)

Next consider second and third terms

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)} \quad \text{or} \quad z\,dy + y\,dz - 2y\,dy = 0$$

on integrating $\boxed{yz - y^2 = c_2}$ — (2)

Hence (1) & (2) Together constitute the solution

Ex 3. $\frac{dx}{y^2} = \frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$

consider $\frac{dx}{y^2} = \frac{dy}{x^2}$ or $x^2 dx = y^2 dy \Rightarrow x^3 - y^3 = c_1$ — (1)

which is first solution

Now consider $\frac{dy}{x^2} = \frac{dz}{x^2y^2z^2}$ Cancelling the common factors, we have

$$\frac{dy}{1} = \frac{dz}{y^2z^2} \Rightarrow y^2 dy = \frac{dz}{z^2}$$

on integrating $\frac{1}{3}y^3 = -\frac{1}{z} + c_2'$ or $y^3 = -\frac{3}{z} + c_2$

$$\Rightarrow \boxed{y^3 + \frac{3}{z} = c_2} \quad \text{— (2)}$$

Eqn (1) & (2) taken together constitute the answers.

Note: Here in this eqn, we could have considered $\frac{dx}{y^2} = \frac{dz}{x^2y^2z^2}$ either and after cancelling y^2, got the eqn $\frac{dx}{1} = \frac{dz}{x^2z^2}$ which would have yielded the solution $x^3 + \frac{3}{z} = c_2$ — (3)

But (2) & (3) are actually the same in the light of soln (1).

Ex solve 1) $\frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{y^2x}$ Ans. $x^3 - y^3 = c_1$ & $x^2 - z^2 = c_2$

2) $\frac{x\,dx}{y^3z} = \frac{dy}{x^2z} = \frac{dz}{y^3}$ Ans: $x^4 - y^4 = c_1$, $x^2 - z^2 = c_2$

3) $\frac{dx}{y} = \frac{dy}{-x} = \frac{dz}{xe^{x^2+y^2}}$ Ans: $x^2 + y^2 = c_1$, $ye^{x^2+y^2} = c_2 - z$

4) $\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$

$$\frac{y}{z} = c_1, \quad \frac{z}{x} - \frac{y^2}{x} + x^2 = c_2$$

Hint: use $y_1 = c_1 z$ to find second solution

5) $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-(x+z)}$ $x = c_1 y$, $\frac{1}{2}xy + yz = c_2$

use $x = c_1 y$ to find second soln.

6. $\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z + \tan(y-3x)}$ $y - 3x = c_1$

$5x = \log[5z + \tan(y-3x)] + c_2$

B] method of multipliers

sometimes we select one or two sets of multipliers say l, m, n or l', m', n' not necessarily constants to find the fourth ratio by which we come to solutions viz if the equation is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

then choose multiplier l, m, n such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR} \quad \text{---- (1)}$$

Now, suppose the choice of l, m, n such that

$$lP + mQ + nR = 0 \Rightarrow l\,dx + m\,dy + n\,dz = 0$$

and if it is exact we may find its integral as

$$F_1(x, y, z) = c_1 \quad \text{-- (2)}$$

Which is the first solution of the system.

If it is further possible to find the other set of multipliers say l', m', n' such that

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$$

also if $l'P + m'Q + n'R = 0$, then $l'dx + m'dy + n'dz = 0$ and solving we get another solution like

$$F_2(x, y, z) = c_2 \quad \text{... (3)}$$

Thus (2) & (3) constitute the solution of the given set of symmetrical equations,

Ex 1. Solve $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$

If we take the first set of multipliers as 1, 1, 1

We have $\dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y} = \dfrac{dx + dy + dz}{1(y-z) + 1(z-x) + 1(x-y)}$

$$= \frac{dx+dy+dz}{0}$$

$\therefore\ dx+dy+dz=0$ and by integration we get $x+y+z=c_1$ — (1)
as first solution,

second set of multipliers may be conveniently chosen as x, y, z
then each ratio =

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{x\,dx+y\,dy+z\,dz}{x(y-z)+y(z-x)+z(x-y)}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$

on integration, it yields $x^2+y^2+z^2=c_2$ — (2)
thus the eqn's (1) & (2) together constitute the required solution of the set.

Ex 2] $$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx}$$

use the multipliers x, y, z & l, m, n.

3) $$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}.$$

use the multipliers $2, 3, 4$ and x, y, z

4) $$\frac{dx}{x(2y^4-z^4)} = \frac{dy}{y(z^4-2x^4)} = \frac{dz}{z(x^4-y^4)}$$

use the multipliers x^3, y^3, z^3 and $\frac{1}{x}, \frac{1}{y}, \frac{2}{z}$.

5) $$\frac{a\,dx}{(b-c)yz} = \frac{b\,dy}{(c-a)xz} = \frac{c\,dz}{(a-b)xy}$$

use set of multipliers x, y, z and ax, by, cz.

6) $$\frac{dx}{y+zx} = \frac{dy}{-x-yz} = \frac{dz}{x^2-y^2}$$

use the set of multipliers $y, x, 1$ and $x, y, -z$

7) $\dfrac{dx}{x^2(y-z)} = \dfrac{dz}{y^2(z-x)} = \dfrac{dz}{z^2(x-y)}$

Use the multipliers $\frac{1}{x^2}, \frac{1}{y^2}, \frac{1}{z^2}$ & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$

8) $\dfrac{dx}{x(y^2+z)} = \dfrac{dy}{-y(x^2+z)} = \dfrac{dz}{z(x^2-y^2)}$

use the multipliers $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ & $x, y, -1$.